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Finding an Efficient Solution to Linear Bilevel Programming Problem: An Effective Approach

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Abstract. Multilevel programming is developed to solve the decentralized problem in which decision makers (DMs) are often arranged within a hierarchical administrative structure. The linear bilevel programming (BLP) problem, i.e., a special case of multilevel programming problems with a two level structure, is a set of nested linear optimization problems over polyhedral set of constraints. Two DMs are located at the different hierarchical levels, both controlling one set of decision variables independently, with different and perhaps conflicting objective functions. One of the interesting features of the linear BLP problem is that its solution may not be Pareto-optimal. There may exist a feasible solution where one or both levels may increase their objective values without decreasing the objective value of any level. The result from such a system may be economically inadmissible. If the decision makers of the two levels are willing to find an efficient compromise solution, we propose a solution procedure which can generate effcient solutions, without finding the optimal solution in advance. When the near-optimal solution of the BLP problem is used as the reference point for finding the efficient solution, the result can be easily found during the decision process.

Keywords: Hierarchical Decision Making, Bilevel Programming, Bicriteria Programming, Effiient Solution.

1. Introduction

Multilevel programming (MLP) is developed to solve the decentralized problem in which decision makers (DMs) are often arranged within a hierarchical administrative structure. The linear bilevel programming (BLP) problem, i.e., a special case of the MLP problems with a two-level structure, is a set of nested linear optimization problems over polyhedral constraints. Two DMs are located at the different hierarchical levels, both of them controlling only one set of decision variables independently, with different and perhaps conflicting objective functions. Control over the decision variables is partitioned between the two levels; however, the decision variables of one level may affect the objective value of the other.

The formal formulation of the linear BLP problem is defined by Candler and Townsly [6], as well as Fortuny-Amat and McCarl [7]. The problem to be considered in this paper has the following common characteristics:

1. There exist interacting decision-making units within a predominantly hierarchical structure.

2. The execution of decisions is sequential, from higher to lower level. The lower-level DM (LLDM) executes its policies after, and in view of, the decisions of the higher-level DM (HLDM).

3. Each decision-making unit optimizes its own objective function independently of other units, but is affected by the actions and reactions of other units.

4. The external effect on a decision-maker's problem can be reflected in both his objective function and his set of feasible decisions.

Let a vector of decision variables $(x, y) \in R^n$ be partitioned among the HLDM and the LLDM. The HLDM controls over the vector $x \in R^{n_1}$ while the LLDM control over the vetor $y \in R^{n_2}$, where $n_1 + n_2 = n$. Furthermore, assuming that the funtion $F, f : R^n \longrightarrow R^1$ are linear and bounded, the linear BLP problem can be stated as follows (see [4], [12]):

$$\begin{aligned} \text{P1:}\quad \max_x\ & F(x,y) = ax + by \\ & \text{where } y \text{ solves} \\ \text{P2:}\quad \max_y\ & f(x,y) = cx + dy \\ \text{s.t.}\ & Ax + By \le r \end{aligned}$$

where $a, c \in R^{n_1}$, $b, d \in R^{n_2}$, $r \in R^m$, A is an $(m \times n_1)$-matrix, and B is an $(m \times n_2)$-matrix. Let $S \subset R^n$ denote the feasible region of (x, y), i.e., $S = \{(x,y) \mid Ax + By \le r\}$. For a given x, let $Y(x)$ denote the set of optimal solutions to the inner problem, P2,

$$\max_{y \in Q(x)} \tilde{f}(y) = dy,$$

where $Q(x) = \{y \mid By \le r - Ax\}$, and represent the HLDM's solution space or the set of rational reactions of f over S, as

$$\psi_f(S) = \{(x,y) \mid (x,y) \in S, y \in Y(x)\}.$$

S and $Q(x)$ are assumed here to be bounded and non-empty. For a given x, the choice of y is reduced to a linear programming problem. The definitions of feasibility and optimality for the linear BLP problem are then given as follows:

DEFINITION 1 *A point (x, y) is called feasible if $(x, y) \in \psi_f(S)$.*

DEFINITION 2 *A feasible point (x^*, y^*) is called optimal if $F(x^*, y^*)$ is unique for all $y^* \in Y(x^*)$ and $F(x^*, y^*) \ge F(x, y)$ for all feasible pair $(x, y) \in \psi_f(S)$.*

The multiple criteria programming (MCP) problem seeks a simultaneous compromise solution among the various goals of different divisions (see [9], [14]). Such technique assumes that all objectives are those of a single DM, or a coherent group of DMs and cannot fully account for the independent behaviour of each division. The bicriteria programming (BCP) problem with only two objectives is a special case of the MCP problems. The linear BCP problem is stated as follows:

$$\begin{aligned} \text{P3: } \max\ & (F(x,y), f(x,y)) \\ \text{s.t. } & Ax + By \leq r. \end{aligned}$$

The notations for the BCP problem are the same as those for the BLP problem in this paper. Some well known definitions and theorems are presented as follows (see [10], [15]):

DEFINITION 3 *A point x is called feasible if $x \in S$.*

DEFINITION 4 *A feasible point x^0 is called efficient or nondominated if there exists no other feasible x, such that $F(x) \geq F(x^0)$ and $f(x) \geq f(x^0)$ with at least one inequality holding.*

THEOREM 1 *A feasible point x^0 is efficient if and only if there exists a $\lambda \in (0,1)$ such that x^0 is optimal for*

$$\begin{aligned} \text{P4: } \max\ & \lambda f(x,y) + (1-\lambda)F(x,y) \\ \text{s.t. } & Ax + By \leq r. \end{aligned}$$

THEOREM 2 *A point x^0 that is the unique solution of maximizing any of the two objective functions subject to S, is efficient.*

One of the interesting features of the linear BLP problem is that its solution may not be efficient (see [1], [5], [12]), i.e., there may exist at least one feasible solution where one or both DMs could increase their objective value without decreasing the objective value of the other level. The result from such a system may be economically inadmissible. For example, in a network design problem described by Ben-Ayed *et al.* [3], the modification of a transportation system is concerned by adding new link or improving existing ones. The objective of the system planner (HLDM) is to minimize total system costs consisting of system travel costs by users (LLDM) and investment costs. However, the objectives of the users are to maximize their individual utility functions, which may conflict with the optimal solution for the system. If the solution of the bilevel network design problem is non-Pareto optimal, the HLDM could maintain the total system costs, such as by increasing investment cost and decreasing the system travel costs, so as to increase the users' individual utility functions and then find an efficient solution.

Under the assumption that the cooperation between both DMs is allowed and both DMs are willing to cooperate when the optimal solution has been found inefficient, Wen and Hsu [13] have presented a post-optimality analysis for obtaining the efficient solution. They suggest several efficient compromise solutions based on the DMs' preference. However, the optimal solution must be identified before the post-optimality analysis, which is quite time-consuming owing to the complexity of the problem (see [2], [11]). In this paper, a solution procedure is proposed to generate efficient conpromise solutions, without finding the optimal solution in advance. The cooperative BLP problem and its related characteristics are described in the next section. In the third section, an efficient approach based on goal setting by DMs is proposed. A numerical example is presented, and concluding remarks are finally made.

2. The Cooperative BLP

When cooperation is allowed and the two DMs are willing to cooperate, the BLP problem turns into the cooperative BLP problem. In the cooperative BLP problem, the concept, coalition, is adopted here which implies that the two DMs become as one in searching for the efficient solution. They maximize the cooperative objective function derived from their original objective functions over a revised constraint region. Before the definition of the cooperative objective function is stated, we assume that payoffs are in monetary terms, utility is linear in money, and interpersonal comparisons are meaningful (see [8]).

Let G_H and G_L be the acceptable objective level of the HLDM and LLDM, respectively, which are the minimum expected goals for both DMs during the cooperative process.

DEFINITION 5 *Let $\lambda \in [0,1]$ be a real value. The objective function of the cooperative BLP problem is*

$$\text{P5:} \quad \Gamma = \lambda(f - G_L) + (1-\lambda)(F - G_H),$$

where $\lambda \in [0,1]$.

Let $f - G_L$ and $F - G_H$ be the bonus functions of both DMs, respectively. Then Γ is the linear combination of both DMs' bonus functions. Restated the cooperation includes determining the weight of each DM's bonus function for appearing in Γ. Actually, this is equivalent to determining the weight of each DM's objective function.

DEFINITION 6 *Let (x_λ, y_λ) be the cooperative efficient solution for the cooperative BLP problem. Consequently, the characteristic function, V, of the cooperative BLP problem is the summation of both DMs' objective values, i.e., $V = F(x_\lambda, y_\lambda) + f(x_\lambda, y_\lambda)$.*

Restated V is the total payoff to the coalition. The characteristic function works only when the payoffs of both DMs are expressed both in monetary terms.

The subsequent problem of the cooperative BLP is the allocation of the payoffs between the members of the coalition, which is called an imputation. This involves the payments that the individual DM may receive. Known as individual rationality, of course no DM will consent to receive less than their own goal values. Before stating the approach to find the cooperative efficient solution, Definitions of side payments and imputations are provided.

DEFINITION 7 *The side payment between the DMs involve exchanging of bonuses between the members of a coalition to equalize any surplus from their cooperation.*

Here, the bonus is equal to the difference between each DM's payoff and his cooperative goal value, i.e., the extra payoff from the cooperation of both DMs.

DEFINITION 8 *If side payments exist, then the bonus to each DM is* $p = (F - G_H + f - G_L)/2$.

DEFINITION 9 *An imputation for the cooperative BLP problem is a vector* $z = (z_1, z_2)$ *satisfying*

(i) $V = z_1 + z_2$

(ii) $z_1 \geq G_H$

$z_2 \geq G_L$,

where z_i *denotes the amount received by the ith DM.* (i) *represents the collective rationality and* (ii) *represents the individual rationality.*

The definitions of side payments and imputations exist only when the payoffs are in monetary terms.

When the optimal solution to the BLP problem is efficient, no cooperative solution that could benefit both DMs more than the optimal solution does. In this case, due to the asymmetric structure of the BLP problem, the cooperative solution is justified under the assumption that satisfying the HLDM is more important than the LLDM. On the other hand, when the optimal solution to the BLP problem is inefficient, two types of approaches are presented to find the cooperative efficient solutions.

Firstly, if the payoffs are expressed in monetary terms but without side payments, then the following parametric LP is used to find the cooperative efficient solution:

$$\begin{aligned} \text{P6:}\ \max\ & \lambda(f - G_L) + (1-\lambda)(F - G_H) \equiv \lambda f + (1-\lambda)F \\ \text{s.t.}\ & Ax + By \le r \\ & F \ge G_H \\ & f \ge G_L, \end{aligned}$$

where $\lambda \in [0,1]$ and $F \ge G_H$ and $f \ge G_L$ are called the goal-value constraints. That is, if the solution to the maximization of Γ over the entire constraint region with goal-value constraints included is (x, y) when λ is given, then the imputation is $(F(x,y), f(x,y))$. Note that in the cooperative BLP problem $\lambda f + (1-\lambda)F$ is equivalent to $\lambda(f - G_L) + (1-\lambda)(F - G_H)$ in P5.

Let S^6 denote the constraint set in P6. The following lemma demonstrates the relationship between the efficient points of S^6 and S.

LEMMA 1 . *Assume that there exists non-degeneracy in P6, then an efficient point to* $\max(F, f)$ *over* S^6 *is also efficient to* $\max(F, f)$ *over* S.

Proof: Let $(\bar{x}, \bar{y})$ be an efficient point to $\max(F, f)$ over S^6. We have

$$\begin{aligned} & F(\bar{x}, \bar{y}) \ge G_H \\ & f(\bar{x}, \bar{y}) \ge G_L. \end{aligned}$$

Suppose $(\bar{x}, \bar{y})$ is not an efficient solution to $\max(F, f)$ over S, then there exists a point $(\hat{x}, \hat{y}) \in S$ such that $F(\hat{x}, \hat{y}) \ge F(\bar{x}, \bar{y})$ and $f(\hat{x}, \hat{y}) \ge f(\bar{x}, \bar{y})$ for at least one inequality holds. This implies that $F(\hat{x}, \hat{y}) \ge G_H$ and $f(\hat{x}, \hat{y}) \ge G_L$. By the definition of S^6, it follows that $(\hat{x}, \hat{y}) \in S^6$. This contradicts with the efficiency of $(\bar{x}, \bar{y}) \in S^6$. ∎

Lemma 2 guarantees that the efficient solutions found in P6 are also efficient to $\max(F, f)$ over S.

Secondly, if the payoffs are expressed in monetary terms and side payments exist, the following parametric LP is used to find the cooperative efficient solution:

$$\begin{aligned} \text{P7:}\ \max\ & \lambda(f - G_L) + (1-\lambda)(F - G_H) \equiv \lambda f + (1-\lambda)F \\ \text{s.t.}\ & Ax + By \le r. \end{aligned}$$

The imputation in this case is $(G_H + p, G_L + p)$. When side payments exist, we have the following lemma:

LEMMA 2 *Assume that there exists non-degeneracy in P7. With side payments existing, only one efficient imputation exists in P7 when* $\lambda = 0.5$.

Proof: Let $p = (F - G_H + f - G_L)/2$. With side payments existing, $(G_H + p, G_L + p)$ is the form for all imputations. Suppose there are at least two efficient solutions, say, when $\lambda = 0.5$ and $\lambda = \bar{\lambda}$. For $\lambda = 0.5$ and $\lambda = \bar{\lambda}$, the corresponding efficient solutions are $(x_{0.5}, y_{0.5})$ and $(x_{\bar{\lambda}}, y_{\bar{\lambda}})$, respectively. But we have $F(x_{0.5}, y_{0.5}) + f(x_{0.5}, y_{0.5}) \geq F(x_{\bar{\lambda}}, y_{\bar{\lambda}}) + f(x_{\bar{\lambda}}, y_{\bar{\lambda}})$. Therefore $p_{0.5} \geq p_{\bar{\lambda}}$. This contradicts with the efficiency of $(x_{\bar{\lambda}}, y_{\bar{\lambda}})$. ■

The important step not yet considered here is goal setting for each DM. The algorithm, including goal setting, is proposed in the following section for finding the cooperative solution.

3. Efficient Solutions for BLP

3.1. Bounds on HLDM

Bialas and Karwan [5] have proposed the algorithm for searching the local optimal solution to the BLP problem. This algorithm basically employs the simplex method for bounded variables as a tool in the solution procedure. Although it can only guarantee the local solution to the BLP problem, it plays a crucial role in its computational efficiency and its employment within algorithms to find the global optimum. This local optimal solution procedure is denoted here by algorithm L.

Let H_{LB} denote the lower bound and H_{UB} denote the upper bound on the objective value of the HLDM. We can find the maximum of cx over the entire constraint region S, and set to H_{UB}, and find the local optimal *via* algorithm L and set its objective value to H_{LB}.

3.2. Bounds on LLDM

Generally speaking, no absolute relationship occurs between the optimal objective values of both DMs (except for some special cases). Now, the bounds on the HLDM are obtained, and by Definition 2, the optimal solution of the BLP problem must be between the lower and the upper bounds of the HLDM. Therefore, the lower and upper bounds are defined respectively, as finding the minimum and the maximum of the LLDM's objective function over the entire constraints with two more bounding constraints by the HLDM, as shown in the following:

$$
\begin{aligned}
\text{P8: } \max\ & f \\
\text{s.t. }\ & Ax + By \leq r \\
& F \leq H_{UB} \\
& F \geq H_{LB}.
\end{aligned}
$$

$$\begin{aligned} \text{P9: } \min\ & f \\ \text{s.t. } & Ax + By \le r \\ & F \le H_{UB} \\ & F \ge H_{LB}. \end{aligned}$$

P8 finds the upper bound, L_{UB}, and P9 provide the lower bound, L_{LB}. The two additional constraints guarantee that the optimal solution of the LLDM would exist between the bounds of the HLDM.

3.3. Goal Setting and Cooperative Efficient Solution

The linear combination of the bounds is used in this section to find the near-optimal solution for each DM. There is no indication that the optimal solution of the HLDM is closer to the HLDM's lower bound or the upper bound. Therefore, the HLDM's upper bound and the lower bound can be simply multiplied by an arbitrary positive value between 0 and 1 and, subsequently, find a near-optimal solution of the HLDM. For the HLDM, however, assuming that the optimal solution is closer to the HLDM's lower bound is reasonable because the lower bound is derived from a feasible solution and such feasible solution is at least a local solution. For the LLDM, insufficient evidence can not verify whether the optimal solution is closer to the LLDM's upper bound or the lower bound. Therefore, for the LLDM, the near optimal solution is estimated from the middle point of the LLDM's bounds.

Let G_H denote the goal value of the HLDM, G_L denote the goal value of the LLDM. Then

$$G_H = \theta H_{LB} + (1 - \theta) H_{UB}, \tag{1}$$

where θ is the arbitrary value and $0 \le \theta \le 1$.

$$G_L = L_{UB}/2 + L_{LB}/2. \tag{2}$$

In (1) we have shown that when θ is setting at 1, it indicates that the lower bound is the optimal solution for the HLDM.

If both DMs have their own goal values and agree to cooperate in order to search for the efficient solution, the cooperative objective function should then be formed. If side payments do not exist, P6 has been proposed which subsequently yields an efficient solution. In P6, $(1 - \lambda)$ and λ denote the weights for the HLDM and the LLDM, respectively. If side payments exist, P7 has been proposed which will provide an efficient solution.

4. A Numerical Example

The efficient solutions generated procedure in previous section is illustrated in the following by a numerical example to describe its results. The results demonstrated in the decision space are depicted in Figure 1.

EXAMPLE 1

$$\begin{aligned}
&P1: \max\ F = -2x + 11y \quad \textit{where } y \textit{ solves}\\
&P2: \max\ f = -x - 3y\\
&\quad s.t.\ x - 2y \leq 4,\\
&\quad\quad 2x - y \leq 24,\\
&\quad\quad 3x + 4y \leq 96,\\
&\quad\quad x + 7y \leq 126,\\
&\quad\quad -4x + 5y \leq 65,\\
&\quad\quad x + 4y \geq 8,\\
&\quad\quad x \geq 0,\\
&\quad\quad y \geq 0.
\end{aligned}$$

Firstly, the maximum value of $-2x + 11y$ over the entire constraint region S is 179.04, and the solution $(\hat{x}, \hat{y}) = (5.3, 17.24)$. Therefore, H_{UB} is set to 179.04. Then we found that the local optimal solution is $(0, 2)$ and the local optimum is 22 by algorithm L. Then we set H_{LB} to 22.

We further solve P8 and P9 as follows:

$$\begin{aligned}
\text{P8: } &\max\ f = -x - 3y\\
\text{s.t. } &(x, y) \in S,\\
&-2x + 11y \geq 22,\\
&-2x + 11y \leq 179.04.
\end{aligned}$$

$$\begin{aligned}
\text{P9: } &\min\ f = -x - 3y\\
\text{s.t. } &(x, y) \in S,\\
&-2x + 11y \geq 22,\\
&-2x + 11y \leq 179.04.
\end{aligned}$$

We have $L_{UB} = -6$ and $L_{LB} = -59.65$. Then the Goal settings for both DMs will be

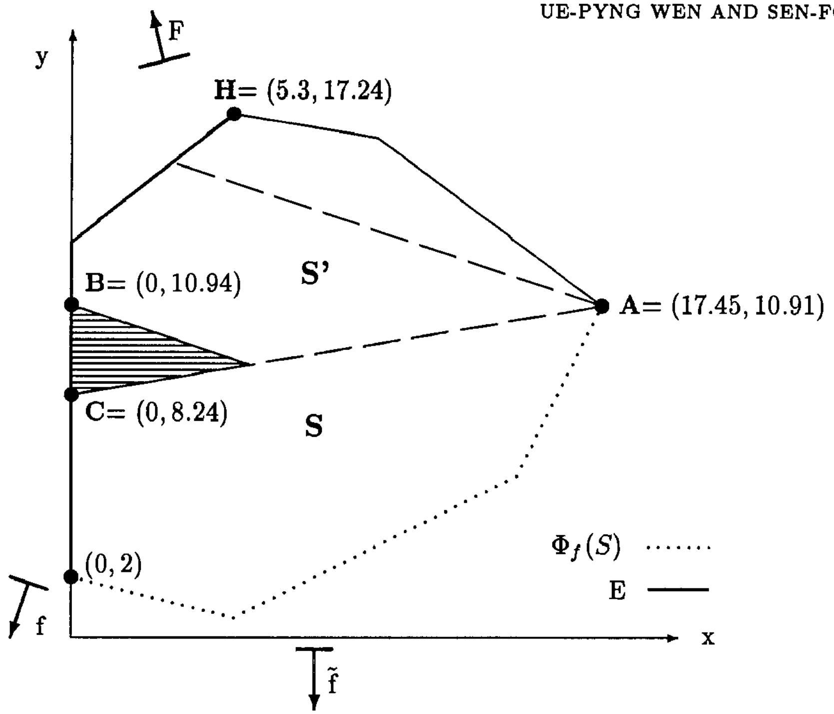

Figure 1. An Illustrative Example

$$G_H = \theta(22) + (1-\theta)(179.04)$$
$$G_L = \tfrac{1}{2}(-6) + \tfrac{1}{2}(-59.65) = -32.83.$$

By selecting $\theta = 0.55$, we have $G_H = 92.67$. Figure 1 shows the revised constraint region (the hatched region) for P6.
Without side payment:

Case 1. Let λ be 0, i.e., we mostly emphasize the bonus for the HLDM. In this case, P6 can be formulated as follows:

$$\begin{aligned} \text{P6: } \max\ & F = -2x + 11y \\ \text{s.t. } & (x, y) \in S, \\ & -2x + 11y \geq 92.67, \\ & -x - 3y \leq -32.83. \end{aligned}$$

The solution to P6 is $B = (0, 10.94)$ and the imputation is $(120.34, -32.82)$.

Case 2. Let λ be 4/5, i.e., the weights for the bonuses of the HLDM and LLDM are 1/5 and 4/5, respectively. In this case, P6 has the following form:

$$\begin{aligned} \text{P6:}\ \max\ & \tfrac{1}{5}(-2x + 11y) + \tfrac{4}{5}(-x - 3y) \\ \text{s.t.}\ & (x, y) \in S, \\ & -2x + 11y \geq 92.67, \\ & -x - 3y \leq -32.83. \end{aligned}$$

The solution to P6 is $C = (0, 8.24)$. The imputation is $(92.67, -25.26)$.

With side payments:
We have P7 as follows:

$$\begin{aligned} \text{P7:}\ \max\ & \tfrac{1}{2}(-2x + 11y) + \tfrac{1}{2}(-x - 3y) \\ \text{s.t.}\ & (x, y) \in S. \end{aligned}$$

By solving P7, we have $H = (5.3, 17.24)$ as the cooperative solution. By Definition 9, the bonus p to each DM is 31.09. The imputation is $(123.76, -1.74)$.

From Figure 1, the optimal solution of the BLP is obtained as Status: RO $A = (17.45, 10.91)$, its objective function vector is $(85.11, -50.18)$ and it is inefficient. S' be the set of points satisfying the following system of inequalities:

$$\begin{aligned} & Ax + BY \leq r, \\ & F(x, y) \geq F(17.45, 10.91), \\ & f(x, y) \geq f(17.45, 10.91). \end{aligned}$$

Furthermore, the cooperative solutions of all the three cases above benefit both the HLDM and LLDM more than A does.

5. Conclusions

The solution to the BLP problem may not be efficient and the existing algorithm takes time in computation for solving the BLP problem. In this study, the near-optimal value of both DMs is used as the goal value which is the minimum level that both DMs would accept when cooperation is allowed. Furthermore, they decide to cooperate in order to find an efficient solution. Having decided the cooperative function, we need to merely solve an LP for finding the cooperative efficient solution.

Further research would find the conditions of existing efficient solution. If the BLP problem can be verified to contain the efficient optimal solution, it would not be necessary for both DMs to cooperate in order to improve their objective function values. Research in real applications by using the proposed algorithm would also be desired. Finally, the cooperation concept could be easily extended to the general multilevel linear programming problem.

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